6. Grundlagen der Integralrechnung

6.1 Geben Sie eine Stammfunktion an:

a) $$\int \left(3x^2 + 2x - 7\right) dx = x^3 + x^2 - 7x$$

b)

$$\int \left(\sqrt{2x} + 3x^{-1/3} + e^{-2x}\right) dx = \int \left(\sqrt{2}x^{1/2} + 3x^{-1/3} + e^{-2x}\right) dx = \frac{2\sqrt{2}}{3}x^{3/2} + \frac{9}{2}x^{2/3} - \frac{1}{2}e^{-2x}$$

c) $$\int (\sin t + \cos t)\, dt = -\cos t + \sin t$$

d) $$\int \left(\frac{1}{x} + \frac{2}{x^2} - \frac{3}{x^3}\right) dx = \int \left(\frac{1}{x} + 2x^{-2} - 3x^{-3}\right) dx = \ln|x| - 2x^{-1} + 6x^{-2}$$

6.2 Integrieren Sie ausführlich mit der linearen Substitution

a)

$$\int \sqrt{2x+8}\, dx = \frac{1}{2}\int \sqrt{z}\, dz = \frac{1}{2}\int (z)^{\frac{1}{2}} dz = \frac{1}{2}\frac{2}{3}(z)^{\frac{3}{2}} = \frac{1}{3}\sqrt{(2x+8)^3} + C$$

$$\text{mit}: z = 2x + 8$$

$$\frac{dz}{dx} = 2 \rightarrow dx = \frac{1}{2}dz$$

b)

$$\int \cos(4t - \pi)\, dt = \frac{1}{4}\int \cos(z)\, dz = \frac{1}{4}\sin(z) = \frac{1}{4}\sin(4t - \pi) + C$$

$$\text{mit}: z = 4t - \pi$$

$$\frac{dz}{dt} = 4 \rightarrow dt = \frac{1}{4}dz$$

c)

$$\int \frac{dx}{3x-5} = \frac{1}{3}\int \frac{dz}{z} = \frac{1}{3}\ln|3x-5| + C$$

$\text{mit}: z = 3x - 5$

$$\frac{dz}{dx} = 3 \rightarrow dx = \frac{1}{3}dz$$

6.3 Berechnen Sie die bestimmten Integrale ausführlich

a)

$$\int_2^6 \frac{3x+3}{3x+2}dx = \int_2^6 \frac{3x+2+1}{3x+2}dx = \int_2^6 1dx + \int_2^6 \frac{1}{3x+2}dx = [x]_2^6 + \left[\frac{1}{3}\ln|3x+2|\right]_2^6$$

$$= 6 - 2 + \frac{1}{3}\left(\ln|3\cdot 6+2| - \ln|3\cdot 2+2|\right)$$

$$\text{NR}: \int \frac{1}{3x+2}dx = \frac{1}{3}\int \frac{1}{z}dz = \frac{1}{3}\ln|3x+2| \qquad = 4 + \frac{1}{3}\ln\left|\frac{20}{8}\right|$$

$$z = 3x+2 \qquad \frac{dz}{dx} = 3 \rightarrow dx = \frac{1}{3}dz \qquad = 4 + \frac{1}{3}\ln\left|\frac{5}{2}\right|$$

b)

$$\int_0^{\pi/3} \sin\left(\frac{t}{2} - \frac{\pi}{6}\right)dt = -2\left[\cos\left(\frac{t}{2} - \frac{\pi}{6}\right)\right]_0^{\pi/3} = -2\left(\cos(0) - \cos\left(-\frac{\pi}{6}\right)\right) = -2\left(1 - \frac{\sqrt{3}}{2}\right) = -2 + \sqrt{3}$$

$$\text{NR}: \int \sin\left(\frac{t}{2} - \frac{\pi}{6}\right)dt = 2\int \sin(z)\,dz = -2\cos\left(\frac{t}{2} - \frac{\pi}{6}\right)$$

$$z = \frac{t}{2} - \frac{\pi}{6} \qquad \frac{dz}{dt} = \frac{1}{2} \rightarrow dt = 2dz$$

c)

$$\int_1^4 \frac{du}{(2u-1)^3} = \left[\frac{-1}{(2u-1)^2}\right]_1^4 = -\frac{1}{7^2} + \frac{1}{1^2} = \underline{\underline{\frac{48}{49}}}$$

NR : $\int \frac{du}{(2u-1)^3} = \frac{1}{2}\int \frac{dz}{(z)^3} = -2\frac{1}{2}z^{-2} = \frac{-1}{(2u-1)^2}$

$z = 2u - 1 \qquad \frac{dz}{du} = 2 \rightarrow du = \frac{1}{2}dz$